Mathematical Exploration

Modelling the Heating Curve

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Introduction

Often I make hot milk for myself when my parents are late from work. To prepare the milk, one must turn on the stove and wait for it to reach a boil and turn off the stove before it overflows. This usually takes a long time and for me this was a problem as I had to waste time standing by the stove waiting for the milk to boil. Another problem was the milk never took the same amount of time to reach a boil on different days as the ambient temperature changes hence it was also not possible to set a timer. Hence, I decided to create a mathematical model to accurately predict the time it takes for milk to reach a boil based on the ambient room temperature which changes all the time.

Data Collection

In order to accurately predict the time it takes for milk to reach a boil I first needed to collect data about how the temperature of milk changes with time on the stove. In order to collect data I brought my small electric stove and sauce pan to school as I was not allowed to borrow the equipment for home. After learning how to do it, I used Vernier's[1] 'Stainless Steel Temperature Probe' along with the 'Logger Pro' software on my computer to take a reading of the temperature of the milk every 10 second as it heated on the electric stove until it reached a boil and started overflowing. The raw data contained 181 reading but I have trimmed the data to only 36 point by only showing every 5th reading for the sake of simplification. The data has been presented below in table 1.

1. [1] Vernier Software & Technology. Logger Pro 3 Quick Reference Guide. http://www2.vernier.com/manuals/LP3QuickRefManual.pdf. Accessed 4 July 2018.

Table 1 – Recorded Data

Time (seconds)	Temperature (°C)
0	22.7
50	26.1
100	30.7
150	34.9
200	38.8
250	42.7
300	46.8
350	50.4
400	54.3
450	57.6
500	61.0
550	64.2
600	66.9
650	70.4
700	72.9
750	75.5
800	77.8
850	79.9
900	82.5
950	84.1
1000	86.0
1050	87.9
1100	89.4
1150	91.0
1200	92.4
1250	93.9
1300	94.8
1350	95.7
1400	96.4
1450	97.0
1500	97.3
1550	97.8
1600	98.1
1650	98.1
1700	98.1
1750	98.1

Method 1 – Digital Curve Fitting

After obtaining these values I used Google Sheets to plot a graph of the data. The scatter plot with blue points represents how the temperature of the milk changes over time on active stove.

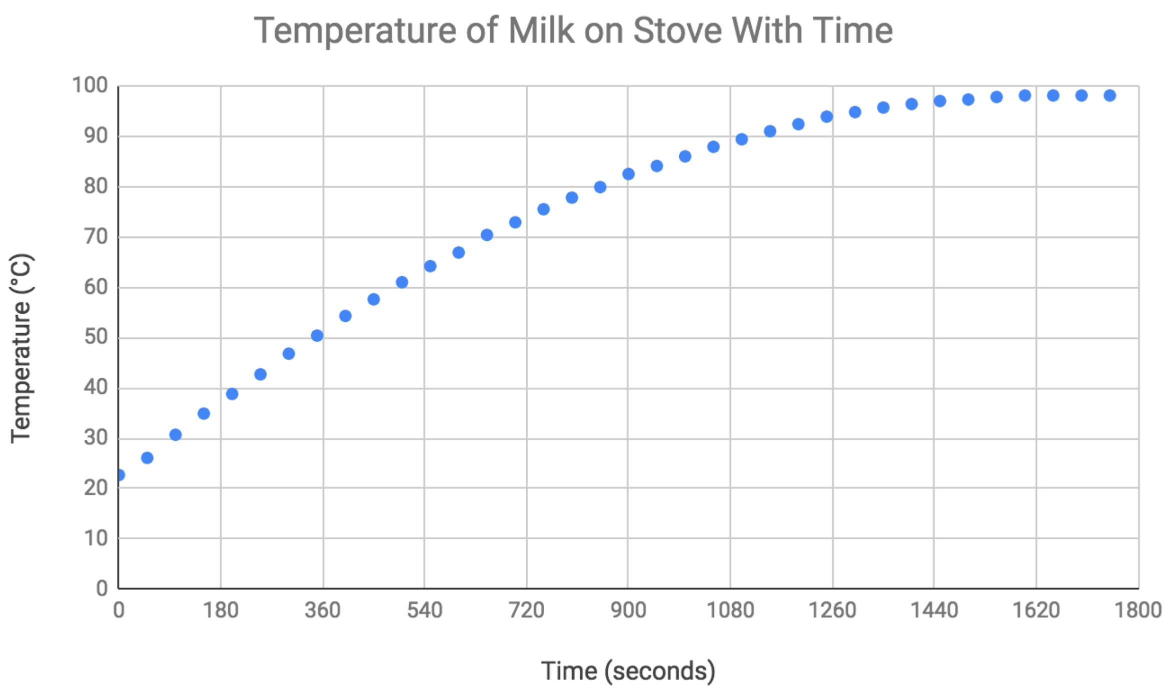

***Figure 1** – actual data graph*

After plotting the data I notice that the graph looks like half part of a parabola hence its mathematical representation would be a quadratic function $ax^2 + bx + c$. I also noticed how the y-intercept represents the ambient room temperature and wondered if it would be possible to use this data in order to model a function which could accurately predict the time required for milk to reach the boiling point which is 98.1 °C at any given initial temperature (ambient room temperature).

I researched how I would be able to use the data in order to model an equation and found out that this method is called 'curve fitting'[2] and it is possible to use 'Google Sheets'[3] to find an equation for a given set of data. After watching some tutorials on how to use this tool I was able to obtain a function and plot a graph (referring to figure 2) where the blue dots represent the actual data and the grey line represents the function $f(x)$ given below.

$$f(x) = -0.0000277x^2 + 0.092x + 21.8$$

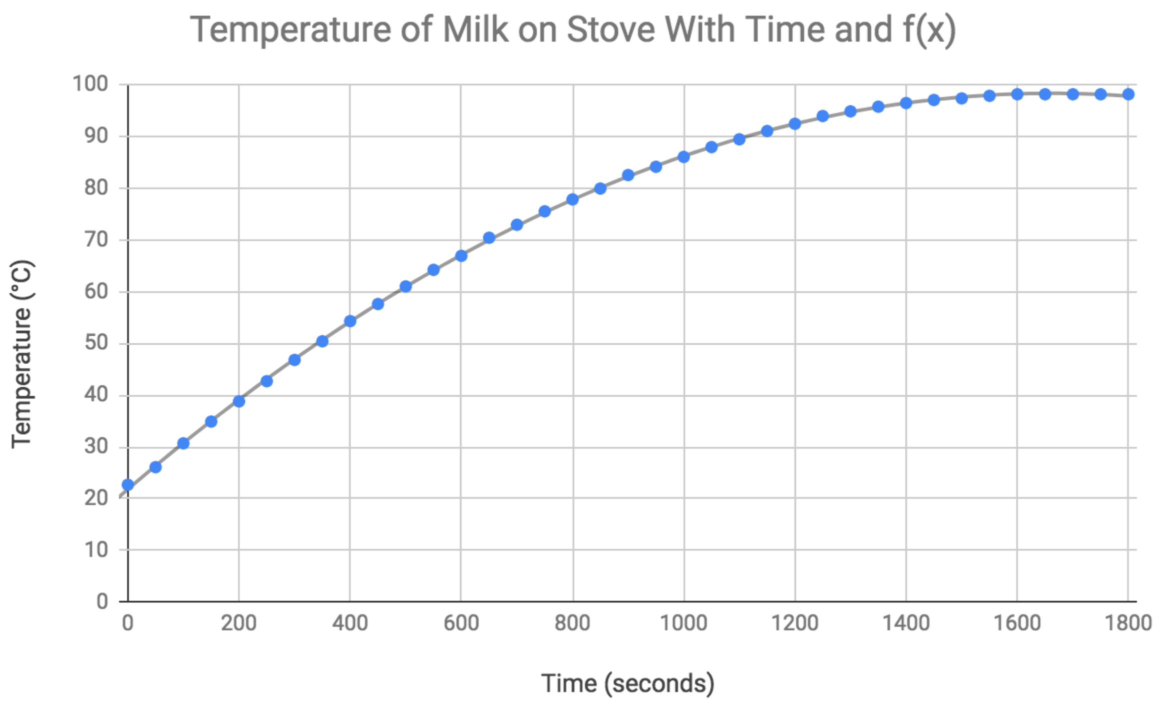

***Figure 2** – actual data and $f(x)$ graph*

2. [2] Sandra Lach Arlinghaus. 1994. CRC Press – Practical Handbook of Curve Fitting.
3. [3] Patricia Goodman. 2 Dec 2014. Add a trendline to a chart in Google Sheets. https://service.goshen.edu/support/index.php?/default_import/Knowledgebase/Article/View/974/0/add-a-trendline-to-a-chart-in-google-sheets. Accessed 10 Jul 2018. Accessed 5 July 18.

The functions fits the actual data with very little uncertainty hence the function can be used to estimate the amount of time required for milk to reach a boil. As my aim is to estimate time with a given temperature, I needed to find an inverse function.

$$f(x) = -0.0000277x^2 + 0.092x + 21.8$$

equating to y

$$y = -0.0000277x^2 + 0.092x + 21.8$$

Dividing the equation by the coefficient of x^2

$$\frac{y}{-0.0000277} = x^2 + \frac{0.092}{-0.0000277}x + \frac{21.8}{-0.0000277}$$

Adding the square of the half of the coefficient of x for making a perfect square

$$\frac{y}{-0.0000277} = x^2 + \frac{0.092}{-0.0000277}x + \frac{21.8}{-0.0000277} + \left(\frac{0.092}{-0.0000554}\right)^2 - \left(\frac{0.092}{-0.0000554}\right)^2$$

Completing the square

$$\frac{y}{-0.0000277} = \left(x + \left(\frac{0.092}{-0.0000554}\right)\right)^2 + \frac{21.8}{-0.0000277} - \left(\frac{0.092}{-0.0000554}\right)^2$$

Squaring the fraction

$$\frac{y}{-0.0000277} = \left(x + \left(\frac{0.092}{-0.0000554}\right)\right)^2 + \frac{21.8}{-0.0000277} - \frac{0.008464}{0.00000000306916}$$

Adding the fractions

$$\frac{y}{-0.0000277} = \left(x + \left(\frac{0.092}{-0.0000554}\right)\right)^2 - 3544761.4331$$

Adding the constant to both sides

$$\frac{y}{-0.0000277} + 3544761.4331 = \left(x + \left(\frac{0.092}{-0.0000554}\right)\right)^2$$

Taking the square root of both sides

$$\pm\sqrt{\frac{y}{-0.0000277}+3544761.4331}=x+\left(\frac{0.092}{-0.0000554}\right)$$

Isolating x

$$\pm\sqrt{\frac{y}{-0.0000277}+3544761.4331}-\left(\frac{0.092}{-0.0000554}\right)=x$$

Writing the function in terms of x and simplifying polarities. Positive values of square root were also removed as they are outside the domain of the actual data. Giving the function $f^{-1}(x)$ below.

$$f^{-1}(x)=-\sqrt{\frac{x}{-0.0000277}+3544761.4331}+\left(\frac{0.092}{0.0000554}\right)$$

Upon graphing the function in Desmos[4], it was confirmed that this is an accurate inverse function as it reflects along the line $y=x$.

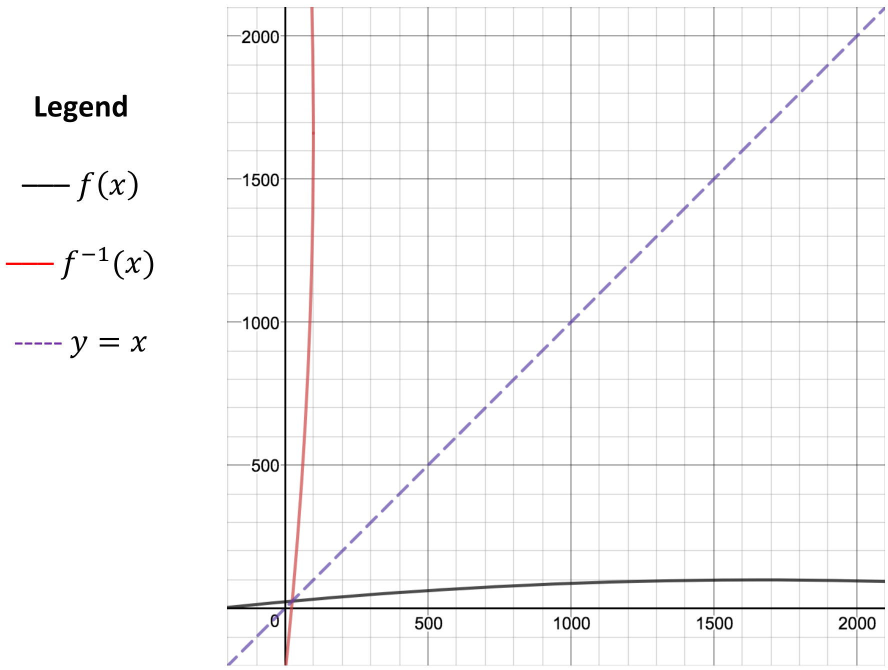

***Figure 3** – $f(x)$ and $f^{-1}(x)$ graph*

4. [4] Desmos. 20 Jul 2018. Functions. https://support.desmos.com/hc/en-us/articles/207316093-Functions. Accessed 21 Jul 18.

In order to make the function valid for any value of initial temperature, I subtracted constant c from x which would represent the difference in ambient temperature compared to the initial temperature in my data which was 22.7 °C. For example, if the room temperature is 35.0 °C, I would subtract $35.0 - 22.7 = 12.3$ to give me the value of c which would be substituted in the below function to translate the entire function horizontally.

$$f^{-1}(x) = -\sqrt{\frac{(x-c)}{-0.0000277} + 3544761.4331} + \left(\frac{0.092}{0.0000554}\right)$$

Hence, using the above function it would be possible to estimate the time required for the milk to reach a certain temperature at any given initial temperature.

Method 2 – Newtons Cooling Law

Newtons law of cooling states that the rate of cooling or heating is proportional to the difference between the temperature of an object and the ambient temperature.[5] In this case, the highest temperature that the milk will reach can be considered the ambient temperature forming the below equation where upper case 'T' represents the temperature and the lower case 't' represents the time. With the help of a video tutorial[6], I was able to form and manipulate the below equation.

$$\frac{dT}{dt} = k(T - 98.1)$$

5. [5] The University of British Columbia. Other differential equations. http://www.ugrad.math.ubc.ca/coursedoc/math100/notes/diffeqs/cool.html. Accessed 4 Aug 2018.
6. [6] Phil Clark. 12 Jan 2016. Differential Equations Newton's Law of Heating Problem. https://www.youtube.com/watch?v=Kqy1xEUnEIE. Accessed 10 Jul 2018.

Both sides divided by $(T - 98.1)$ and multiplied by dt

$$\frac{dT}{T - 98.1} = kdt$$

This is a differential equation in the variable separable format. Hence, to solve:

Both sides integrated

$$\int \frac{dT}{T - 98.1} = \int kdt$$

New function formed after integration

$$\ln(T - 98.1) = kt + C_1$$

Both sides written as powers of e

$$\mathrm{e}^{\ln(T\,-\,98.1)} = e^{kt+C_1}$$

Power of e removed and new constant 'a' introduced

$$T - 98.1 = ae^{kt}$$

Adding 98.1 to LHS and RHS

$$T = 98.1 + ae^{kt}$$

Finding the value of a using the value of T at t = 0 from data.

At t = 0, T = 22.7 as per data

$$22.7 = 98.1 + a$$

Subtracting 98.1 from both sides

$$a = -75.4$$

Forming new equation with the value of a

$$T = 98.1 - 75.4e^{kt}$$

Subtracting 98.1 from both sides

$$T - 98.1 = -75.4e^{kt}$$

Dividing both sides by 75.4

$$\frac{T-98.1}{-75.4} = e^{kt}$$

Finding the natural log of both sides

$$\ln\frac{T-98.1}{-75.4} = kt$$

Dividing both sides by k

$$k = \frac{\ln\frac{T-98.1}{-75.4}}{t}$$

In order to obtain an accurate value of k, I decided to use several values of temperature (T) and time (t) from my data. I used Google Sheets to use the above equation for k to calculate the value of k for all of the data. The average of these values was later calculated to be used as the final k.

***Table 2** – value of k for every 4^{th} data point*

Time (t)	Temperature (T)	Value of k
0	22.7	N/A
200	38.8	-0.001200989845
400	54.3	-0.001357933644
600	66.9	-0.001357933644
800	77.8	-0.001640232986
1000	86.0	-0.001829601822
1200	92.4	-0.002151950917
1400	96.4	-0.002708699303
1600	98.1	N/A
Average		-0.001749620309

The value of k was also rounded off to 5 decimal places

$$T = 98.1 - 75.4e^{-0.00175x}$$

Giving the final function $g(x)$

$$g(x) = 98.1 - 75.4e^{-0.00175x}$$

Just as I did with the previous method, in order to obtain values of time with temperature, I would need to find an inverse function.

Function made equal to y in order to calculate inverse

$$y = 98.1 - 75.4e^{-0.00175x}$$

98.1 subtracted from both sides

$$y - 98.1 = -75.4e^{-0.00175x}$$

Both sides divided by -75.4

$$\frac{y - 98.1}{-75.4} = e^{-0.00175x}$$

Natural log of both sides taken

$$\ln\frac{y - 98.1}{-75.4} = -0.00175x$$

Both sides divided by -0.00175

$$x = \frac{\ln\frac{y - 98.1}{-75.4}}{-0.00175}$$

Final function written in terms of x

$$g^{-1}(x) = \frac{\ln\frac{x - 98.1}{-75.4}}{-0.00175}$$

Again, just like the previous equation, constant c was subtracted from the equation which would represent the difference of room temperature from 22.7, the actual room temperature from the data used to model the function. This makes the function valid to calculate the time required to reach a boil at any given value of room temperature.

$$g^{-1}(x) = \frac{\ln \frac{(x-c)-98.1}{-75.4}}{-0.00175}$$

Evaluation

Using the above 2 methods I have modelled the data in 2 different ways. Below, Table 3 and the graph in figure 4 compares the data against the actual collected set of data.

***Table 3** – modelled functions against actual data and mean difference*

Temp	Time	$f^{-1}(x)$		$g^{-1}(x)$	
Data		Value	Difference	Value	Difference
22.7	0	9.81	9.81	0	0
26.1	50	47.42	2.58	26.37	23.63
30.7	100	99.73	0.27	64.09	35.91
34.9	150	149.08	0.92	100.86	49.14
38.8	200	196.4	3.6	137.26	62.74
42.7	250	245.29	4.71	176.13	73.87
46.8	300	298.58	1.42	220.07	79.93
50.4	350	347.16	2.84	261.64	88.36
54.3	400	401.89	1.89	310.38	89.62
57.6	450	450.14	0.14	355.15	94.85
61.0	500	501.95	1.95	405.25	94.75
64.2	550	552.92	2.92	456.8	93.2
66.9	600	597.82	2.18	504.22	95.78
70.4	650	659.03	9.03	572.21	77.79
72.9	700	705.14	5.14	626.26	73.74
75.5	750	755.59	5.59	688.49	61.51
77.8	800	802.69	2.69	749.82	50.18
79.9	850	848.07	1.93	812.22	37.78
82.5	900	908.04	8.04	900.31	0.31
84.1	950	947.45	2.55	962.14	12.14
86.0	1000	997.27	2.73	1045.49	45.49

87.9	1050	1051.16	1.16	1143.1	93.1
89.4	1100	1097.33	2.67	1233.99	133.99
91.0	1150	1151.18	1.18	1350.12	200.12
92.4	1200	1203.46	3.46	1475.62	275.62
93.9	1250	1267.11	17.11	1650.13	400.13
94.8	1300	1310.82	10.82	1787.93	487.93
95.7	1350	1360.84	10.84	1969.91	619.91
96.4	1400	1406.45	6.45	2166.96	766.96
97.0	1450	1453.39	3.39	2415.71	965.71
97.3	1500	1481.41	18.59	2597.69	1097.69
97.8	1550	1542.01	7.99	3158.16	1608.16
98.1	1600	1603.68	3.68	N/A	N/A
Mean Difference			4.86		246.56

Legend

Actual Data

$$f^{-1}(x) = -\sqrt{\frac{x}{-0.0000277} + 3544761.4331} + \left(\frac{0.092}{0.0000554}\right)$$

$$g^{-1}(x) = \frac{\ln\frac{x-98.1}{-75.4}}{-0.00175}$$

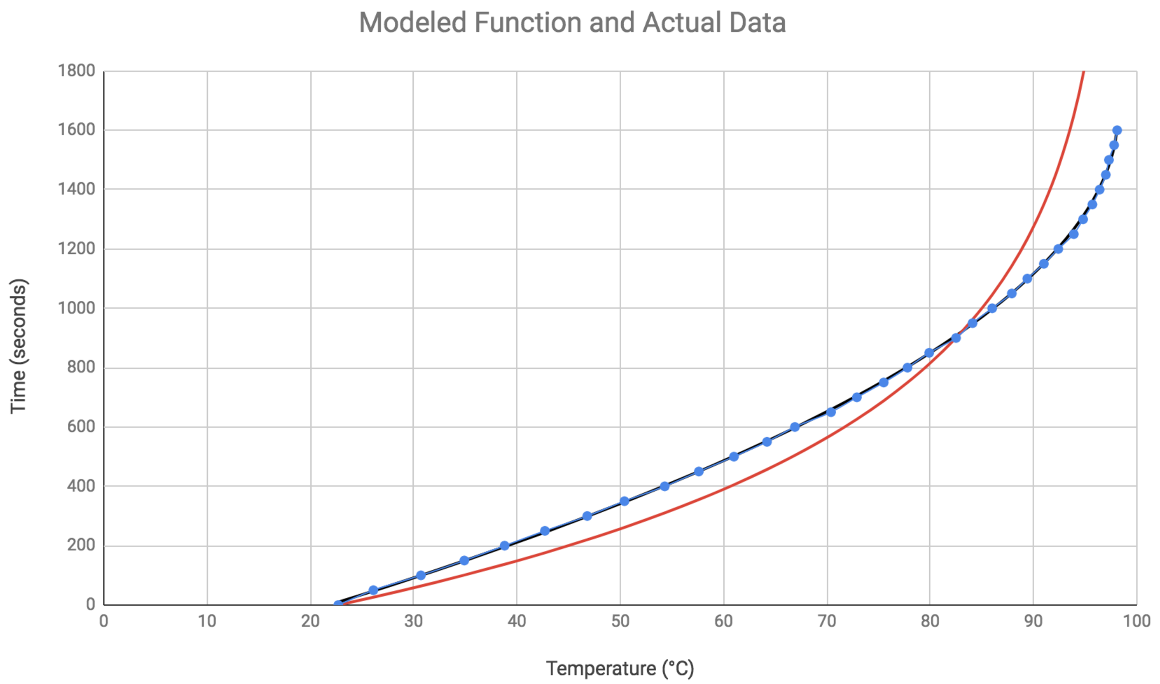

***Figure 4** – actual data against modeled functions*

Conclusion

As the mean difference and the graph clearly indicates, the quadratic model $f^{-1}(x)$ is clearly a more accurate model. Both of these models allow for the subtraction of constant c from x making both models useful in calculating the time required for milk to reach boiling point for any given initial temperature.

In order to accurately estimate the time it would take for milk to reach a boil in the particular stove, I would calculate the value of c based on the room temperature and substitute the boiling point of milk which is 98.1 in the more accurate function $f^{-1}(x)$.

For example, taking the current room temperature of my kitchen which is 30.0 °C, I can calculate the value of $c = 30.0 - 22.7 = 7.3$. Then, Input the value of the boiling point of milk which is 98.1 into function $f^{-1}(x)$.

$$f^{-1}(x) = -\sqrt{\frac{(x - 7.3)}{-0.0000277} + 3544761.4331} + \left(\frac{0.092}{0.0000554}\right)$$

$$f^{-1}(98.1) = -\sqrt{\frac{(98.1 - 7.3)}{-0.0000277} + 3544761.4331} + \left(\frac{0.092}{0.0000554}\right)$$

$$f^{-1}(98.1) = -\sqrt{\frac{90.8}{-0.0000277} + 3544761.4331} + \left(\frac{0.092}{0.0000554}\right)$$

$$-\sqrt{\frac{90.8}{-0.0000277} + 3544761.4331} + \left(\frac{0.092}{0.0000554}\right) = 1144$$

To the nearest second, it would take 1144 seconds for my milk to reach a boil when the room temperature is 30.0 °C in the place where the stove is kept.

Limitations and Further Scope

Limitations

I believe one of the main limitations of my exploration was the limitations of my experiment. For example, temperature fluctional in the room I conducted my experiment in could have contributed to inaccurate data and hence resulting in an inaccurate mathematical model. In order to reduce inaccuracies induced by variations in room temperature, the experiment can be conducted in a more insulted environment for example the recording room in our school which is has foam walls providing good thermal insulation and minimizing room temperature variations. One other limitation of my experiment was the lack of trials conducted, conducting more trials would have reduced random uncertainty in the experiment data.

I think the limitation of the model I created was its restriction to be only useful on a certain type of stove heating at a certain rate. This is because the rate at which the stove is heating the liquid would change the heating curve making the function useless for accurately predicting the time on a different stove. I think my model can be greatly improved is by modifying the function in such way that the intensity of the heat source can be accounted for in calculating time. Modifying the function in a such a way would make it much more useful as it would be possible for one to modify the function based on the type of stove they are using increasing the scope of its application.

Further Application

Other than determining the time it takes milk to reach a specific temperature on a stove, the same method can be applied to different liquids in order to model their heating curve. This can be useful for industrial applications as factories can easily determine the time a certain heating process will take. The same can also be applied in restaurants where automatic temperature tracking instruments are usually not available.

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